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Large-N corrections to the strong coupling behaviour of $\mathrm{SU}(N)/\mathbb{Z}_N$ lattice gauge theories

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Abstract: We derive a formula for the large N behaviour of the expectation values of an arbitrary product of Wilson loops in the adjoint representation. We show the consequences of our formula for the study of the large N strong coupling behaviour of $\mathrm{SU}(N)/\mathbb{Z}_N$ pure gauge theories, and theories with matter fields in the adjoint representation. This allows us to calculate large N corrections to the gluino condensate and meson propagators in the lattice version of Supersymmetric Yang-Mills. Applications to the strong coupling behaviour of the Kazakov-Migdal model are also given.

Keywords: Strong Coupling Expansion, 1/N Expansion, Lattice Gauge Field Theories.

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Contents

1. Introduction

SU(N) group integration formulas at large N have attracted interest since the late 70's. They occur in the study of large N lattice gauge theories and in connection with matrix models. The latter have found a wide range of applications over the years in areas going from 2-dimensional quantum gravity to 4-dimensional Yang-Mills theory. The class of matrix models involving large N group integrations are the unitary matrix models (see ref. [1] for a recent review).

Within lattice gauge theories, SU(N) group integrations arise naturally when studying strong coupling expansions of lattice gauge theory models, which simplify in the large N limit. The most standard method [2] uses an effective action approach centered around the one-link integral. Our motivation for this paper arose in the context of $\mathcal{N} = 1$ SUSY Yang-Mills on the lattice, an interesting testing ground for non-perturbative methods, and for which some Monte Carlo data are available [3]. The model contains lattice gauge fields and quarks (gluinos) in the adjoint representation of the group. When studying the model at strong coupling one notices that the effective potential method cannot be carried through, lacking an equivalent one-link integral formula in the adjoint representation. Our approach [4] was based on the hopping parameter expansion expressed as a sum of random walks on the lattice [5, 6]. Then, one has to perform the gauge field integration of an expression containing products of traces of Wilson loops in the adjoint representation. It turns out, as will be shown in this paper, that the leading large N contribution is rather trivial from the point of view of large N integration. It arises from loops which are pure-spikes (or pure backtrackers) and that actually do not depend on the gauge fields.

Another instance in which group integrals of quantities in the adjoint representation have acquired relevance is in the context of *induced QCD* [7]. The Kazakov-Migdal model has scalar quarks in the adjoint representation coupled to a $\beta = 0$ lattice gauge field. A wide variety of techniques have been applied to this model. The strong coupling (large mass) region solution of the gaussian model was obtained by Gross [8]. Again, as realised later by Khoklachev and Makeenko [9], the role of the large N integration over the gauge variables is simply to restrict the sum over random walks to pure-spike paths. Their contribution does not depend on the gauge fields, so that it can be taken outside the integral.

In this paper we will address the problem of computing the leading non-trivial large N integral of a product of traces of Wilson loops in the adjoint representation. The result is more easily presented in the form of the evaluation of a generating function (eq. (2.1)) for adjoint Wilson loops. The main result is presented in eq. (2.13). The formula is obtained by expanding the exponential, performing the integration and resumming the result, so that it is only valid for sufficiently small lattice couplings $s(\gamma)$, i.e. it is the strong coupling expression of the generating functional. The result is the exponential of a quantity which is of order 1 ($\sim N^0$). This contrasts with the typical N^2 behaviour for the fundamental representation. Indeed, in applications to strong coupling expansions of theories with matter fields, there is an additional $N^2 - 1$ term arising from trivial Wilson loops which are proportional to the trace of the unit matrix in this representation. This term comes from the pure-spike paths mentioned previously. Thus, the terms we are calculating here give large N subleading contributions to the observables of these theories. We emphasise nevertheless that our calculation gives the leading term in which the actual details of the group integration play any role.

The paper is organised as follows. The main quantity we are calculating is given in eq. (2.1) of section 2, and the result for it is given in eq. (2.13). The rest of the section contains the outline of the derivation, with details and terminology collected in the two appendices (this includes a useful diagrammatic formalism to deal with the computation of group invariants). Readers not interested in the derivation might jump directly to section 3, where the formula in question is put to work in calculating large N corrections to the strong coupling values of observables in $\mathrm{SU}(N)/\mathbb{Z}_N$ lattice gauge theories, with or without matter fields. Explicit formulas are given for the lattice version of $\mathcal{N} = 1$ Supersymmetric Yang-Mills theory. This includes the large N corrections to the gluino condensate and the leading non-OZI contributions to the scalar meson propagator. Similar calculations can be carried on for the gaussian Kazakov-Migdal model. As an example, we show how from our formulas one can deduce the leading N behaviour of the propagators of the p-scalar bound states. The masses of these states are explicitly given. Finally, the paper is ended with a short conclusion section.

2. Group integration formulas

We consider an SU(N) gauge theory defined on a hypercubic, d-dimensional infinite lattice Λ. To any link ℓ of the lattice there is associated a group element g_ℓ. Most expressions that we will use involve the matrices corresponding to the adjoint representation of the group $\mathcal{U}_A(g_\ell) \equiv \mathcal{U}_\ell$. Regarding geometrical objects on the lattice, such as lattice paths, we will adopt the terminology and conventions of ref. [6]. Given a closed lattice path γ, $\mathrm{Tr}(W_A(\gamma))$ labels the trace of the corresponding Wilson loop evaluated in the adjoint representation.

Our main goal in this section is to compute, to leading order in N, the result of integrating over the group variables g_ℓ an arbitrary product of traces of adjoint Wilson loops. This result is best collected in the form of the following generating function:

$$\mathcal{Z}(s) = \prod_\ell \left(\int d\mu_H(g_\ell) \right) \exp \left\{ \sum_{\widetilde{\gamma}} s(\widetilde{\gamma})\ \mathrm{Tr}(W_A(\widetilde{\gamma})) \right\} . \tag{2.1}$$

In this expression, $d\mu_H(g_\ell)$ is the Haar measure on the group variable g_ℓ, and the product extends to all the links ℓ of the lattice. The sum in the exponent runs over all equivalence classes of closed paths. Each equivalence class $\widetilde{\gamma}$ contains all paths such that the traces of the adjoint Wilson loops computed along them coincide for an arbitrary gauge field configuration. The common value of the trace of the adjoint Wilson loop is labelled $\mathrm{Tr}(W_A(\widetilde{\gamma}))$. The arguments of the generating function are the real parameters $s(\widetilde{\gamma})$. Notice that $\mathcal{Z}(s)$ can also be looked at as the partition function of a certain kind of pure gauge theory. The parameters $s(\widetilde{\gamma})$ would in that case appear as the (lattice) couplings of the corresponding action. Although the action is not the most general SU(N)/$\mathbb{Z}_N$ invariant action, other terms would give superfluous information. From this point of view the result of this section can be interpreted as the evaluation of the partition function of this pure gauge model in the strong coupling region (small values of $s(\widetilde{\gamma})$).

The expansion of the exponential in $\mathcal{Z}$ gives us the integrals of traces that we are interested in:

$$\mathcal{Z}(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\widetilde{\gamma}_1} \cdots \sum_{\widetilde{\gamma}_n} s(\widetilde{\gamma}_1) \cdots s(\widetilde{\gamma}_n) \mathcal{E}(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n) \, , \tag{2.2}$$

with

$$\mathcal{E}(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n) = \prod_\ell \left(\int d\mu_H(g_\ell) \right) \mathrm{Tr}(W_A(\widetilde{\gamma}_1)) \cdots \mathrm{Tr}(W_A(\widetilde{\gamma}_n)) \, . \tag{2.3}$$

To compute the previous expression one must first select a representative of the class $\widetilde{\gamma}$. We will choose a closed path without spikes (backtracking parts). It is always possible to reduce an arbitrary path by eliminating its spikes in a systematic way and reach a path which is called *simple* in ref. [6]. As we will see later there is more than one simple path within each class, but anyone will do. Let us, from now on, use the same symbol $\widetilde{\gamma}$ to refer to the class and to its representative simple path. In the rest of this section we will show the main steps in the evaluation of expression (2.3). Many of the technical details are collected in appendices A and B.

A possible approach would be to express the adjoint Wilson loops in terms of fundamental representation ones and then use the formulas available for the latter. Instead we would use a procedure which could allow the calculation of traces of adjoint Wilson loops and other gauge invariant objects systematically to all orders in $1/N$. What we do first is to express the integrand in eq. (2.3) as a sum of products of the elements of the adjoint representation matrices $\mathcal{U}_\ell$:

$$\mathcal{E}(\widetilde{\gamma}_1,\ldots,\widetilde{\gamma}_n)=\sum_{a_i,b_i}\delta(\rho)^{a_1\ldots a_L b_1\ldots b_L}\prod_\ell\left(\int d\mu_H(g_\ell)\right)\mathcal{U}_{\ell_1}^{a_1b_1}\cdots\mathcal{U}_{\ell_L}^{a_Lb_L}\,. \tag{2.4}$$

The sum extends over the $2L(N^2-1)$ possible values of the indices. For simplicity we will refer to the indices jointly by the set $\mathcal{I}\equiv\{a_1,b_1,\ldots a_L,b_L\}$. The integer L gives the length of the total joint path $\widetilde{\gamma}_1\ldots\widetilde{\gamma}_n$. The labels $\ell_1\ldots\ell_L$ refer to the links of this joint path taken in an arbitrary order and where a given link can occur more than once in the sequence. Finally, the symbol $\delta(\rho)$ contains the contraction of the indices necessary to produce the traces given in eq. (2.3). It is given by a product of L Kronecker deltas. It is clear that which index is contracted with which depends on the naming of the indices and on the geometry of the paths. This information can be coded into an application $\rho:\mathcal{I}\to\mathcal{I}$ which maps the contracted indices onto each other ($\rho(c)=c'\,;\rho(c')=c$). Since the application is bijective it can be seen as an element of the group $\mathbf{S}_{2L}$ of permutations of $2L$ indices, having the properties of a *pairing*. It is clear from the definition of the index set $\mathcal{I}$, that there is another pairing $\tau\in\mathbf{S}_{2L}$ naturally defined on our system. It is the application which maps the two indices of a given link matrix onto each other: $\tau(a_i)=b_i$, $\tau(b_i)=a_i$.

The next step in the calculation is the evaluation of the group integrals involved in eq. (2.4). Our procedure will be to express the adjoint representation matrices in terms of the fundamental representation ones, and then to make use of the available formulas for that case. Here we will just explain and make use of the results, collecting the details of the derivation within appendix A. First, one can look at the result of integrating over a single link variable. The result can be expressed as a linear combination of invariant group tensors, with coefficients depending on N. These tensors have indices within those elements of $\mathcal{I}$ which belong to the matrices corresponding to this link. The result of integrating over all link variables can be expressed similarly as a linear combination of invariant tensors, this time involving all the indices of the set $\mathcal{I}$. Thus we can write:

$$\mathcal{E}(\widetilde{\gamma}_1,\ldots,\widetilde{\gamma}_n)=\sum_{c_i}\delta(\rho)^{c_1\ldots c_{2L}}\sum_{\vartheta\in\mathcal{S}_\Gamma}C_N(\vartheta)\,\mathcal{T}(\vartheta)^{c_1\ldots c_{2L}}\,, \tag{2.5}$$

where $\mathcal{T}(\vartheta)^{c_1\ldots c_{2L}}$ is an invariant tensor and $C_N(\vartheta)$ the corresponding N-dependent coefficient. The indices c_i simply label the elements of $\mathcal{I}$. The symbol ϑ stands for

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a permutation ($\vartheta : \mathcal{I} \to \mathcal{I}$). In appendix A, eq. (A.11), we describe how one can associate to any permutation of a certain type an invariant tensor ($\vartheta \longrightarrow \mathcal{T}(\vartheta)$). There, one can also see that if the permutation has a nonvanishing number of 1-cycles, the corresponding tensor vanishes. It is clear by the construction that both the invariant tensor and its coefficient factorise into a product of contributions of single links. This and other properties (like the absence of 1-cycles) restrict the space of possible ϑ. The full set labelled $\mathcal{S}_\Gamma$ defines a certain subgroup of $\mathbf{S}_{2L}$. We will leave out its characterisation for later and continue with the outline of the procedure.

The final step consists in performing the summation over the values of the indices, which is equivalent to contracting the colour indices of the invariant tensor with the $\delta(\rho)$. Defining

$$D_N(\vartheta,\rho) \equiv \sum_{c_i} \delta(\rho)^{c_1\ldots c_{2L}}\, \mathcal{T}(\vartheta)^{c_1\ldots c_{2L}} \tag{2.6}$$

we finally get:

$$\mathcal{E}(\widetilde{\gamma}_1,\ldots,\widetilde{\gamma}_n) = \sum_{\vartheta\in\mathcal{S}_\Gamma} C_N(\vartheta) D_N(\vartheta,\rho)\,. \tag{2.7}$$

The result is expressed in terms of the coefficients $C_N(\vartheta)$ and $D_N(\vartheta,\rho)$. Explicit forms for these quantities can be traced back from eqs. (A.15) and (A.16) in appendix A. The main point now is to analyse the behaviour of these coefficients as $N \to \infty$.

The large-N behaviour of the coefficients $C_N(\vartheta)$ follows from the result for the corresponding coefficient in the one-link integral, eq. (A.9); one gets:

$$C_N(\vartheta) = A(\vartheta)\ \left(\frac{2}{N}\right)^L +\ \text{subleading terms}\,. \tag{2.8}$$

where the leading prefactor $A(\vartheta)$ is zero for all but a restricted class of permutations ϑ. It is not hard to characterise this subset within $\mathcal{S}_\Gamma$. It is given by those permutations ϑ satisfying: $\vartheta\tau\vartheta\tau = \mathsf{e}$, where e is the identity mapping. For these permutations the coefficient $A(\vartheta)$ is equal to 1. This result rests on the work of ref. [10].

The machinery to deal with the contractions of invariant tensors entering the definition of D_N, and extract their leading N behaviour, is set up in appendix B, whither we refer for details. The method follows by setting up a Feynman-diagram interpretation of each term in the sum eq. (2.7). For that we need to enter into the structure of the invariant tensors $\mathcal{T}(\vartheta)$. These are products of traces of products of the group generators in the fundamental representation (cf. eq. (A.13)). To any p-cycle contained in the permutation ϑ there corresponds an invariant tensor of the form $\mathrm{Tr}(\lambda^{c_1}\cdots\lambda^{c_p})$ (where λ^{c_i} is a group generator). It is clear why the tensor vanishes if there are 1-cycles. We look at these traces as vertices of a Feynman diagram where the indices play the role of the legs. The vertices are not symmetric with respect to the exchange of the legs but only cyclic-symmetric. This form of vertices is characteristic of planar diagrams in large hermitian matrix models, for example. Thus, for the full ϑ one obtains a bunch of vertices with a total of $2L$ legs. The contractions encoded in ρ can be seen as propagators joining pairs of legs of different or the same vertices. Thus to every term in eq. (2.7) one can associate a (vacuum) Feynman diagram with V vertices and L lines (note that L, which is the total number of pairs of colour indices in eq. (2.4), is the only topological quantity of the diagrams being equal for all the ϑ and ρ). We further call K the number of connected components of the diagram.

In appendix B we prove that the large N maximal leading contribution of a diagram with K connected components, V vertices, and L lines is given by $m\,2^{-L}N^{2K-V+L}$, where m is an integer that can be zero or one. Corrections go like inverse powers of N^2. If we combine this with (2.8), we obtain that the leading term coming from $C_N(\vartheta)D_N(\vartheta,\rho)$ goes at most as N^{2K-V}.

Now, from the topological constraints it is clear that $1 \leq K \leq V$. Therefore, the maximum value of $2K - V$ would occur when $K = V$. This means that all propagators go from one vertex to itself. To understand the implications of this we need to make use of some of the properties of the vertices appearing in such a diagram and its relation to the original set of lattice paths. Since vertices arise from link integration they can only contain indices pertaining to matrices of the same link. Furthermore, vertices only mix the first indices of the matrices among themselves and the second among themselves (see appendix A for details). Hence, a contraction between two indices of the same vertex comes from a contraction of the link matrix with itself. But since the first index is contracted with the first or the second with the second, we have either $\mathcal{U}_\ell^t\mathcal{U}_\ell$ or $\mathcal{U}_\ell\mathcal{U}_\ell^t$, which due to the orthogonal nature of the adjoint representation matrix allows to eliminate the two matrices altogether. In terms of the original paths a self contraction corresponds to the path moving in one direction forward and backward consecutively. This gives what we call spikes. But remember that our original paths were simple, and hence had no spikes. In conclusion, the restriction to simple paths tells us that there are no self-contractions of our diagrammatic vertices, corresponding to the situation when these vertices come from normal ordered operators.

We then conclude that all connected components of the diagram have at least two vertices. Hence, the leading term arises from diagrams for which $2K = V$ and which give a contribution of order one as $N \to \infty$. Finally, we arrive at

$$\mathcal{E}(\widetilde{\gamma}_1,\ldots,\widetilde{\gamma}_n) = \nu_0 + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{2.9}$$

where ν_0 is an integer summing the contribution from all the leading order diagrams. This is the main result that was needed in ref. [4] to justify the restriction to *pure-spike* paths. In what follows we will proceed to compute the value of ν_0.

First of all let us study what can we learn about the graphs giving nonvanishing leading contribution ($\mathcal{O}(N^0)$). We have not yet given the condition that has to be satisfied by ϑ to give a nonvanishing leading contribution to $D_N(\vartheta, \rho)$. This occurs (see appendix B) when the permutation ϑ satisfies the relation: $\vartheta\rho\vartheta\rho = \mathsf{e}$. This is very similar to the characterisation coming from the link integration, with the replacement of τ by ρ. Our final number ν_0 comes from counting the number of possible permutations ϑ within $\mathcal{S}_\Gamma$ satisfying both relations simultaneously. Furthermore, the set $\mathcal{S}_\Gamma$ is characterised by saying that it contains all permutations having no 1-cycles and such that all cycles contain indices belonging to matrices of the same link and to either the first or the second index of these matrices (but not both).

To understand the implications of the previous conditions, let us return to the connection between the original paths $\widetilde{\gamma}_i$ and the permutation ρ. Any of the paths is an ordered sequence of matrices. Since the matrices have two indices we get a classification of indices into two sets: those appearing as a first or a second index of the matrices. Notice that if the link is followed in the reverse direction the definition of what index is first or last is exchanged. Any path is then associated to two equal length closed sequences of indices. It corresponds to two of the cycles of the permutation $\rho\tau$. The two cycles can be connected by means of τ (or ρ). Now, from the conditions characterising the leading ϑ, we conclude:

$$\rho\tau = \vartheta\rho\tau\vartheta^{-1}\,. \tag{2.10}$$

This means that ϑ must map complete cycles of $\rho\tau$ into equal length cycles. Given the identification of these cycles with the original paths, we see that a leading ϑ just maps complete paths of equal length among themselves. But since ϑ only connects indices belonging to the same lattice point, we conclude that the actual paths connected by ϑ have to coincide. There are a few very simple conclusions one can extract from this result. The first is that there is a factorisation property satisfied by the integral $\mathcal{E}(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n)$. Its value factorises into a product of contributions of distinct paths. This follows from a similar factorisation property for ϑ, since indices from different paths cannot be connected by it, despite the fact that the paths might overlap partially. Thus, the final number of maximal permutations ν_0 factorises as well. One might have guessed this factorisation from the standard arguments in the large N limit; notice, however, that the factorisation is not complete: as we will see, the integral of a given Wilson loop to the power n is not the n-th power of the integral of a single Wilson loop, so that the result is not classical.

It only remains, therefore, to compute ν_0 for the case of a single Wilson loop $\mathrm{Tr}(W_A(\widetilde{\gamma}))$ taken to the power n. To do this, it is convenient to express the path $\widetilde{\gamma}$ in the form $(\gamma')^\omega$, where ω is a positive integer. This notation means that $\widetilde{\gamma}$ is built up by describing γ' sequentially ω times; γ' itself must not be reducible equivalently. We will refer to ω as the *winding number* of the path, and write it as $\omega(\widetilde{\gamma})$. These considerations are necessary, as our results will depend explicitly on $\omega(\widetilde{\gamma})$.

For our path configuration we have that $\rho\tau$ has n cycles of length $l(\widetilde{\gamma})$ (the length of the path). The permutation ϑ maps one cycle into other preserving the ordering within each cycle. Thus, it is specified by giving the corresponding mapping of the indices at a given space point. Naively one would say that the number is then $n!$. In fact this has to be corrected by two facts: the first is that we have to eliminate all permutations ϑ having 1-cycles; the second is that one must take into account that if $\omega(\widetilde{\gamma})$ is larger than 1, the mapping has to respect the cyclic ordering in which $\widetilde{\gamma}$ describes these points. This last issue can be dealt with by arranging the indices belonging to the same path into cycles of length $\omega(\widetilde{\gamma})$. In a compact mathematical form we have that, given a permutation τ' having n ω-cycles, we want to count how many permutations ϑ' without 1 cycles of the $n\omega$ indices satisfy: $\vartheta'\tau'\vartheta'^{-1} = \tau'$. Let us call this number P^n_ω. The action of ϑ' is simply the mathematical statement of a change of names of the indices, so that without the requirement of no 1-cycles, the solution would be simply $n!\omega^n$. This comes because we must map complete cycles into complete cycles, so we have $n!$ possibilities. Then after having selected the mapping of the cycles, there are ω possible assignements for each cycle (it is enough to specify the mapping of 1 element of the cycle and the rest would follow): this gives the ω^n. Subtracting the permutations which have 1-cycles takes a little more effort but presents no real problem. As a matter of fact, it is easier and more compact to determine the expression for the generating function first and deduce from it the expression of the quantity P^n_ω. For the generating function corresponding to a single path one obtains:

$$\prod_\ell \left(\int d\mu_H(g_\ell) \right) \exp\{s(\widetilde{\gamma})\ \mathrm{Tr}(W_A(\widetilde{\gamma}))\} = \frac{\exp\{-s(\widetilde{\gamma})\}}{(1-\omega(\widetilde{\gamma})s(\widetilde{\gamma}))}\,, \tag{2.11}$$

For completeness we give an expression for the numbers P^n_ω themselves:

$$P^n_\omega = n! \sum_{p=0}^{n} \frac{(-1)^p}{p!}\,\omega^{n-p}\,. \tag{2.12}$$

Finally, we combine the result eq. (2.11) for all the paths via the factorisation property and arrive at the desired leading contribution to the generating functional eq. (2.1):

$$\mathcal{Z}(s) = \exp\left(-\sum_{\widetilde{\gamma}} \left(s(\widetilde{\gamma}) + \log(1-\omega(\widetilde{\gamma})s(\widetilde{\gamma}))\right) \right), \tag{2.13}$$

which is the main result of this section. As explained in the introduction the exponent is of order N^0. Had we added a contribution to the initial action proportional to the trace of the unit matrix in the adjoint representation, the term would have remained unchanged by the integration (being a constant) and would have given a contribution to the exponent of order N^2-1 (the dimension of the adjoint representation). Although trivial, this remark is important in practice because such a term actually arises in many applications. It can be associated to the trivial path of zero length.

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3. Application to strong coupling expansions

In this section we will use the group integration formula obtained in the last section to derive results which are relevant for the behaviour of $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theories on the lattice, with or without matter fields, at strong coupling and large N. These include pure gauge theories, gauge theories with fermions in the adjoint representation, a particularly interesting case being the lattice version of $\mathcal{N}=1$ Supersymmetric Yang-Mills [3], and also gauge-Higgs models with scalars in the adjoint representation like the Kazakov-Migdal model [7]. In all cases, after integrating out the matter fields via a hopping parameter expansion, one ends up with an effective action of the type given in eq. (2.1). Our integration formula eq. (2.13) allows us to write immediately the expression for the free energy in the strong coupling region at large N. In what follows we will give the explicit formulas in some of the cases that have been studied previously in the literature.

Consider first the case of $\mathrm{SU}(N)/\mathbb{Z}_N$ pure gauge lattice theories. For this case the consequence of our formulas can be easily deduced. As mentioned at the beginning of section 2, the generating function $\mathcal{Z}(s)$ can be seen as the partition function of a pure gauge model where the couplings $\beta_A(\widetilde{\gamma})$ are identified with $s(\widetilde{\gamma})$. The free energy of the model is then given by the exponent of the right hand side of eq. (2.13). As a particular example we can write down the case of the Wilson-type plaquette action model. The free energy per unit volume of the model is given by:

$$\frac{F(\beta_A)}{\mathrm{Volume}} = \frac{d(d-1)}{2}\left(-\beta_A - \log(1-\beta_A)\right), \tag{3.1}$$

which has a singularity at $\beta_A = 1$. Actually, the position of the singularity coincides with the position of the third-order phase transition point in the 2-dimensional lattice gauge theory [11]. In higher dimensions the model is known to have a first order phase transition line at $\beta_A < 1$ [12, 11]. Thus, the strong coupling phase depicted by eq. (3.1) extends into the metastable region as is usually the case for these transitions. There is one atypical behaviour of the free energy eq. (3.1) which explains why this formula was not found before. This is the fact that it is of order $1 = O(N^0)$. Typically the free energy goes like N^2, as for example is the case in this same model above the first order phase transition point. The different N dependence of the free energy above and below the transition suggests that the position of the transition is given by the vanishing of the weak coupling free energy. This is indeed the case in 2 dimensions. An N^2 behaviour is also shown by the free energy of the mixed fundamental-adjoint action for any non-zero value of the fundamental coupling [11]. Thus, in older work the free energy (3.1) of this model is taken to be zero [13]. Remarkably, from eq. (2.13) one can easily read out the free energy for an arbitrary $\mathrm{SU}(N)/\mathbb{Z}_N$ invariant action.

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Let us now consider the addition of matter fields in the adjoint representation. The general features and behaviour is similar for scalar or fermionic matter. For definiteness we will illustrate the formulas restricting to the fermionic case. Indeed, our concern with $\mathcal{N}=1$ SUSY Yang-Mills triggered our interest in the problem. Here we will follow the procedure of ref. [4]. Performing the integration over the quark (gluino) fields leads to a determinant. In the large mass (small hopping) region, this determinant can be written as the exponential of an effective action expressable as a sum over closed paths on the lattice:

$$S_{eff} = \eta_f \sum_{x\in\Lambda} \sum_{l=1}^{\infty} \sum_{\gamma\in\mathcal{S}_l(x)} \frac{1}{l} \operatorname{Tr}(W(\gamma)) \operatorname{Tr}(\Gamma(\gamma)) \,, \tag{3.2}$$

where $\mathcal{S}_l(x)$ is the set of closed paths of length l with origin in the lattice point x, and $\Gamma(\gamma)$ is a spin matrix associated to the path, whose explicit form we will detail later (cf. eq. (3.5)). The factor η_f takes into account the different possibilities for the nature of the fermion field: $\eta_{\text{Majorana}} = 1/2$, while $\eta_{\text{Dirac}} = 1$.

In ref. [6] it is shown how it is convenient and possible to rearrange the summation over paths into a sum over classes of paths labelled by what is called a *simple path*. A simple path is a closed contour that has no spikes or backtracking pieces. It is easy to see that for all paths associated with the same simple path the trace of the Wilson loop is exactly the same, so that they actually belong to a single equivalence class of the type defined at the beginning of section 2. However, the classes considered in this paper contain more than one simple path and its corresponding classmates. The reason is that in ref. [6] closed paths are defined as ordered sets of links, which have therefore a first and a last element. It is clear, however, that which link is taken as the first one is irrelevant for the gauge field dependence, being given by a trace. This is just a technical point, but essential to get the numbers right. To compensate for this point one has simply to multiply by a counting factor which gives the number of links that can be chosen as first element. Curiously this factor is not the length of the path $l(\widetilde{\gamma})$ but $l(\widetilde{\gamma})/\omega(\widetilde{\gamma})$, where $\omega(\widetilde{\gamma})$ is the winding number factor defined in the last section. When the path goes round a given loop $\omega(\widetilde{\gamma})$ times, it is clear that the relevant factor is the number of links in the basic loop, and not in the total path. In addition, the orientation of the path is irrelevant for the gauge field dependence, so that our present classes contain each path and its reverse. In summary, we stress that the classes of paths defined in section 2 contain a basic simple path, its reverse and all the $l(\widetilde{\gamma})/\omega(\widetilde{\gamma})$ simple paths obtained by selecting a point along the path (as origin).

Finally, one can rewrite the effective action (eq. (3.2)) in the form of the exponent of eq. (2.1), with coefficients $s(\widetilde{\gamma})$ which depend on the basic parameters of the original theory. These coefficients are $\mathcal{O}(1)$ in N. One can include in them the contribution of the pure gauge action. Restricting for definiteness to one flavour adjoint fermions at zero gauge couplings, which contains the lattice version of $\mathcal{N}=1$ supersymmetric

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Yang-Mills, one obtains the following coefficients:

$$s(\widetilde{\gamma}) = \frac{\tau(\widetilde{\gamma})}{\omega(\widetilde{\gamma})}, \tag{3.3}$$

$$\tau(\widetilde{\gamma}) \equiv \frac{\eta_f}{(1-\xi)^{l(\widetilde{\gamma})}} \left(\mathrm{Tr}(\Gamma(\widetilde{\gamma})) + \mathrm{Tr}(\Gamma(\widetilde{\gamma}^{-1}))\right), \tag{3.4}$$

where γ^{-1} is the reverse of the path γ, and the matrix $\Gamma(\widetilde{\gamma})$ is the spin matrix of the path, given by the product along the path of the link matrices:

$$\kappa\ (r\,\mathbb{1}_s - \gamma_\alpha)\,. \tag{3.5}$$

In the previous expression κ is the hopping parameter, r the Wilson parameter, and the spin matrices γ_α are just the Dirac matrices up to a sign, positive if the path traverses the link with positive orientation and negative if not. The factor $1-\xi$ in eq. (3.4) is the result of resumming all paths within each class, with ξ given by:

$$\xi = \frac{1-\sqrt{1-4(2d-1)\kappa^2(r^2-1)}}{2}, \tag{3.6}$$

where d is the space-time dimension.

With the previous expression and formula (2.13), one can obtain the free energy of the model at large N:

$$F = \eta_f\,(N^2-1)\,F_0 \text{ Volume} - \sum_{\widetilde{\gamma}} \left(\frac{\tau(\widetilde{\gamma})}{\omega(\widetilde{\gamma})} + \log(1-\tau(\widetilde{\gamma}))\right), \tag{3.7}$$

where the sum runs over the equivalence classes of paths $\widetilde{\gamma}$ defined in section 2.

We see that our result provides the $\mathcal{O}(1)$ contribution to the free energy. This is not however the leading piece, which is given by the term proportional to F_0, and is of order N^2. The latter was given in ref. [6] and its value is:

$$F_0 = \eta_f \left(-d\log(1-\xi) + (d-1)\log\left(1-\frac{2d}{2d-1}\xi\right)\right) \mathrm{Tr}(\mathbb{1}_s)\,, \tag{3.8}$$

where $\mathbb{1}_s$ is the identity matrix in spin space. This term arises from the paths whose gauge field contribution is proportional to the trace of the unit matrix. This occurs for the so-called pure-spike paths, which are paths such that after eliminating all backtracking parts we are left with a single point (a path of zero length). This contribution is, however, fairly trivial from the point of view of the group integration, so that the $\mathcal{O}(1)$ contribution is the first one in which the characteristics of group integration show up.

Obtaining a closed expression for the $\mathcal{O}(1)$ contribution to the free energy involves performing the sum over path classes explicitly. Similar resummations can be performed (see ref. [6] for a derivation of these resummations and for earlier references on the subject). The appearance of the factor $\omega(\widetilde{\gamma})$ in the formula has prevented us so far from obtaining such a closed expression.

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Other interesting quantities can be computed to the order in N at which we are working. This includes the gluino condensate, which can be obtained by an appropriate differentiation of the free energy. As in the previous case, the leading term given in ref. [4] comes from pure-spike paths, and is (up to some multiplication factors) independent of the group representation in which the quarks are living. This is a particular manifestation of the triviality from the point of view of group integration of the leading term. Finally, our expression is as follows:

$$\begin{aligned}\langle\bar{\Psi}(x)\Psi(x)\rangle = &-\frac{(N^2-1)\,\mathrm{Tr}(\mathbb{1}_s)}{1-\frac{2d}{2d-1}\xi} - \\ &-\frac{1}{\mathrm{Volume}}\sum_{\widetilde{\gamma}}\frac{l(\widetilde{\gamma})\,\tau(\widetilde{\gamma})}{\eta_f\,(1-2\xi)}\left(-\frac{1}{\omega(\widetilde{\gamma})}+\frac{1}{1-\tau(\widetilde{\gamma})}\right),\end{aligned} \tag{3.9}$$

where $l(\widetilde{\gamma})$ is the length of any representative simple path. The division by the volume in the previous expression is simply accounted for by picking a single simple path $\widetilde{\gamma}$ from the space of all space-time translates. The correction is the first unquenched contribution. The leading one coincides with the result of the quenched approximation [4]. To arrive at expression eq. (3.9) we have made use of the resummation formulas of ref. [6]. The origin of the $1-2\xi$ factor can be traced back to the rearrangement of sum over paths into sums over simple paths.

In a similar fashion, it is possible to derive the large N correction to the meson and p-gluino propagators and spectrum, which are given in ref. [4]. We will not give the formulas explicitly since they are notationally fairly lengthy. Instead we will provide the expression of a quantity for which our present computation gives the leading large N behaviour. This quantity is the non-OZI contribution to the meson propagator. In a theory with different equal mass flavours, this term gives the main contribution to the mass difference between the non-singlet and singlet mesons. It contains what in the continuum would be the contribution of the anomaly. When expanding the fermion propagators in terms of paths, the non-OZI contributions come from 2 closed paths originating at the 2 meson positions (see ref. [4] for details). Due to the main factorisation property which we have obtained for Wilson loop integration at this order in N, the contribution is only non-zero when the two paths (classes) coincide. To the order in N at which we are working, this term is non-zero both in the quenched approximation and in the full theory. For simplicity, we give the contribution to the scalar propagator:

$$G_{non-OZI}(x,y) = \frac{1}{\eta_f^2\left(1-\frac{2d}{2d-1}\xi\right)^2}\sum_{\hat{\gamma}_1\hat{\gamma}_2}\tau(\hat{\gamma}_1)\tau(\hat{\gamma}_2)\,\delta_{\widetilde{\gamma}_1\widetilde{\gamma}_2}\left(\frac{\omega(\widetilde{\gamma})}{1-\tau(\widetilde{\gamma})}\right)^2, \tag{3.10}$$

where the sum now extends over all closed terse paths $\hat{\gamma}_1$ with origin in x and $\hat{\gamma}_2$ with origin in y (see ref. [6] for an explanation of the lattice path terminology and an account of the resummation formulas). The $\delta_{\widetilde{\gamma}_1\widetilde{\gamma}_2}$ imposes that the two paths have the same common reduced simple path, which we label $\widetilde{\gamma}$. It is possible to eliminate the constraint given by the δ in the previous formula, but its treatment exceeds the purpose of this paper.

In a similar way one can study other models, like the Kazakov-Migdal model which has scalars in the adjoint representation. The gaussian model can be treated in a similar way as the fermionic model. In this model there are composites formed by p scalars, which are the analog of the p-gluino bound states of the fermionic model [4]. The operators which produce these states can be written in our notation as follows:

$$\mathbf{O}_p(x) \equiv \phi_{a_1}(x) \cdots \phi_{a_p}(x)\, \mathcal{T}(\mathsf{L}_+^{(p)})^{a_1 \ldots a_p} \,. \tag{3.11}$$

At strong coupling the correlator of two operators at different points (say x and y) can be written as a sum over p paths $(\gamma_1, \ldots, \gamma_p)$ joining the fields at the two different points. The gauge field dependence of such a contribution is as follows (we restrict to correlators that are leading at large N):

$$\mathcal{T}(\mathsf{L}_+^{(p)})^{a_1 \ldots a_p}\ \mathcal{T}(\mathsf{L}_+^{(p)})^{b_p \ldots b_1}\, (W_A(\gamma_1))^{a_1 b_1} \cdots (W_A(\gamma_p))^{a_p b_p} \,. \tag{3.12}$$

Now we have to evaluate its expectation value with respect to the gauge field in the large N limit. The quantity is not a product of traces of Wilson loops. To handle this in a simple way with our formulas, we might add a new *big link* $\bar{l}$ going from y to x and its corresponding group element (as an independent variable). Now, one can form p closed paths by attaching to each of the original open paths γ_i the big link $\bar{l}$ (note they all have $\omega = 1$). One can then compute the expectation value of the product of the corresponding Wilson loops. Clearly s for these paths is zero, since none of the paths in the effective action contains the big link. The large N leading term of the connected expectation value is obtained for all paths belonging to the same class. The corresponding contribution is the p-th derivative of the free energy in eq. (2.13) at $s = 0$, which is $(p-1)!$. What has this to do with the expectation value of eq. (3.12)? To see that, one has simply to integrate over the big link group variable. One gets terms like eq. (3.12) multiplied by $(2/N)^p$. Then one has to multiply by the number of different p-cycles, which is $(p-1)!$. Hence, finally, the expectation value of eq. (3.12) to leading order is given by $(N/2)^p$, and is achieved when all the path classes $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_p$ coincide. Similar results can be obtained in the fermionic model, thus justifying the treatment of p-gluinos of ref. [4]. As an application of this result one can proceed in a similar fashion as in ref. [4] and deduce the values of the masses M_p of the p-field bound states for this theory:

$$\cosh(M_p) = \frac{(2d-1)^{p/2}}{2} \left(\left(\frac{1-\xi}{\xi}\right)^{p/2} + \frac{1}{(2d-1)^{p-1}} \left(\frac{\xi}{1-\xi}\right)^{p/2} \right) - \sigma \,, \tag{3.13}$$

σ being an integer number in $\{-(d-1), \ldots, (d-1)\}$, which accounts for all the possible doubler modes (see [4] for details), and ξ is given by the same formula as before (eq. 3.6) with $r^2 - 1$ set to 1.

4. Conclusions

In this paper we have studied the leading behaviour for large number of colours of SU(N) group link integrals of products of traces of Wilson loops in the adjoint representation. We have proven that the leading (non-trivial) result is $\mathcal{O}(1) = \mathcal{O}(N^0)$. The corresponding coefficient has been computed in section 2. The result factorises into contributions of Wilson loops associated to inequivalent paths. This allows us to compute up to this order and in the strong coupling (small $s(\gamma)$) region the corresponding generating function, given in eq. (2.13) of section 2.

In section 3 we have shown how this generating functional and its derivatives enter in expressions which are relevant in the context of strong coupling expansions for gauge theories with matter fields in the adjoint representation. We have given the expression for the free energy of a pure gauge SU(N)/$\mathbb{Z}_N$ model at strong coupling. Adjoint Wilson loop expectation values can be read out immediately for these theories. We then proceeded to show how the large mass expansion of theories with matter fields in the adjoint representation yields an effective action suitable for treatment with our formula. We illustrate the applications by calculating explicitly the large N corrections to various quantities in the lattice version of Supersymmetric Yang-Mills. In all cases, to obtain a compact form, one is faced with a path resummation problem which at the moment we have been unable to solve. (The problem looks tractable, though). We also showed that our present formulas can be used to prove the form of the leading large N behaviour for p-gluino propagators and spectra given in ref. [4]. We then took the case of the gaussian Kazakov-Migdal model to show how, similarly, one arrives at the large N values of the masses of the p-particle bound states at strong coupling.

From the methodological point of view, our paper might also prove of interest in other contexts where the large N limit is at play. Our diagrammatical technique is non-standard and could help to complement other well-known techniques as loop equations, orthogonal polynomials, master field equation, etc. It allows the computation of higher order corrections in $1/N^2$ keeping the Feynman diagram analogy. On the negative side, such a technique does not look appropriate for dealing with the weak coupling (large $s(\gamma)$) behaviour of the generating function.

Acknowledgments

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A. Adjoint (multi)link integration formulas

In this appendix we study group integrals with variables in the adjoint representation of the gauge group. For this purpose, as explained in section 2, we will relate the matrices in the adjoint representation to their fundamental counterparts, and afterwards we will use known integration formulas for the latter. These will be first applied to the computation of single link integrals, and then we will use the result to evaluate the multilink integral in eq. (2.4), and reduce it explicitly to the compact form in eq. (2.5).

Let us first establish some notations and conventions. We write λ^a for the generators of SU(N) in the fundamental representation. They close the group algebra:

$$[\lambda^a, \lambda^b] = \imath f^{abc}\lambda^c\,, \tag{A.1}$$

where the indices a, b, c range from 1 to $N^2 - 1$. We choose the generators to be normalised as:

$$\mathrm{Tr}(\lambda^a\lambda^b) = \frac{1}{2}\,\delta^{ab}\,. \tag{A.2}$$

Furthermore, the completeness relation is written as:

$$(\lambda^a)^{ij}(\lambda^a)^{kl} = \frac{1}{2}\,\delta^{jk}\delta^{il} - \frac{1}{2N}\,\delta^{ij}\delta^{kl}\,. \tag{A.3}$$

Now, the group integrals we want to compute have the general form:

$$\int d\mu_H(g)\,\mathcal{U}^{a_1b_1}\cdots\mathcal{U}^{a_nb_n}\,, \tag{A.4}$$

where $\mathcal{U} = \mathcal{U}_A(g)$ is the matrix corresponding to a given group element g in the *adjoint* representation, and $d\mu_H(g)$ is the Haar measure on SU(N).

The relation of $\mathcal{U}$ with the matrix $U = \mathcal{U}_F(g)$ corresponding to the same group element in the fundamental representation reads, with the conventions adopted above:

$$\mathcal{U}^{ab} = 2\ \mathrm{Tr}(U^\dagger\lambda^aU\lambda^b)\,. \tag{A.5}$$

This allows us to rewrite (A.4) as an integral for matrices in the fundamental representation, whose indices are contracted with a certain tensor built up of a product of generators. Before casting this statement into an explicit equality, let us introduce the following known formula for group integrals in the fundamental representation [10]:

$$\int d\mu_H(g)U^{i_1j_1}\cdots U^{i_nj_n}(U^\dagger)^{l_1m_1}\cdots(U^\dagger)^{l_nm_n} = \sum_{\sigma,\sigma'\in\mathbf{S}_n}\delta(j,\sigma(l))\delta(m,\sigma'(i))\,C_N^n(\sigma\circ\sigma')\,,\qquad \text{with } n < N\,. \tag{A.6}$$

In this expression, $\mathbf{S}_n$ is the permutation group of n elements, and we have introduced the following notation:

$$\delta(j,\sigma(l)) \equiv \prod_{k=1}^{n}\delta_{j_kl_{\sigma(k)}}\,, \tag{A.7}$$

where $l_{\sigma(k)}$ is the k-th element of $\sigma(\{l_1,\ldots,l_n\})$, with $\sigma\in\mathbf{S}_n$.

The coefficients C_N^n are class-functions on the space of permutations, i.e. C_N^n is a map $\mathbf{S}_n\to\mathbb{R}$ fulfilling $C_N^n(\sigma\circ\sigma'\circ\sigma^{-1})=C_N^n(\sigma')$ for any (σ,σ') (the notation $\sigma\circ\sigma'$ stands for the composition of two permutations), so that the value $C_N^n(\sigma)$ only depends on the total number of cycles in the permutation σ. For this purpose we call $\mathcal{C}_k(\sigma)$ the number of k-cycles in σ, and $\mathcal{C}(\sigma)\equiv\sum_k\mathcal{C}_k(\sigma)$. With this notation, it can be shown that the following relation holds [10]:

$$\delta_{\sigma'\tilde{\sigma}}=\sum_{\sigma\in\mathbf{S}_n}C_N^n(\sigma'\circ\sigma)\,N^{\mathcal{C}(\sigma\circ\tilde{\sigma})}\,, \tag{A.8}$$

which allows to determine $C_N^n(\sigma)$ recursively. Its general form turns out to be $C_N^n(\sigma)=P(N)/Q(N)$, where $Q(N)=N^2(N^2-1)\cdots(N^2-(n-1)^2)$, and $P(N)$ is a polynomial whose leading term is $N^{\mathcal{C}(\sigma)}$. Hence, at large N we get:

$$C_N^n(\sigma)\overset{N\to\infty}{=}\frac{1}{N^n}\delta_{\sigma,\mathsf{e}}+\text{ subleading terms}\,, \tag{A.9}$$

where e is the identity permutation, which maps a sequence to itself.

If we now merge eqs. (A.5) and (A.6), the result for (A.4) can be written as:

$$\int d\mu_H\,\mathcal{U}^{a_1b_1}\cdots\mathcal{U}^{a_nb_n}=2^n\sum_{\sigma,\widetilde{\sigma}\in\mathbf{S}_n}C_N^n(\sigma\circ\widetilde{\sigma})\,\mathcal{T}(\sigma)^{a_1\ldots a_n}\,\mathcal{T}(\widetilde{\sigma})^{b_1\ldots b_n}\,, \tag{A.10}$$

where we have introduced the following notation:

$$\mathcal{T}(\sigma)^{a_1\ldots a_n}\equiv(\lambda^{a_1})^{j_1k_1}\cdots(\lambda^{a_n})^{j_nk_n}\delta(k,\sigma(j))=(\lambda^{a_1})^{j_1j_{\sigma(1)}}\cdots(\lambda^{a_n})^{j_nj_{\sigma(n)}}\,, \tag{A.11}$$

an implicit summation over matrix indices being understood. These objects are products of traces of products of generators, and therefore invariant group tensors. To make this more explicit, we can introduce the one-step-forward cyclic permutation of m elements $\mathsf{L}_+^{(m)}$, defined by $\mathsf{L}_+^{(m)}(a_i)=a_{i+1}$ for $i<m$ and $\mathsf{L}_+^{(m)}(a_m)=a_1$, such that:

$$\mathcal{T}(\mathsf{L}_+^{(m)})^{a_1\ldots a_m}=\mathrm{Tr}(\lambda^{a_1}\cdots\lambda^{a_m})\,. \tag{A.12}$$

Then we notice that each cycle in a generic permutation of n elements σ, as that appearing in (A.11), closes an L_+ trace, so that we can write:

$$\mathcal{T}(\sigma)^{a_1\ldots a_n}=\prod_{i=1}^{\mathcal{C}(\sigma)}\mathcal{T}(\mathsf{L}_+^{(\nu_i)})^{a_{\sigma_i(1)}\ldots a_{\sigma_i(\nu_i)}}\,, \tag{A.13}$$

where σ_i is a sub-permutation giving the index order inside the i-th cycle, which is a ν_i-cycle.

Now we address the computation of eq. (2.4). If we substitute for each single-link integration in it the result (A.10) we get:

$$\mathcal{E}(\widetilde{\gamma}_1,\ldots,\widetilde{\gamma}_n) = 2^L \sum_{a_i,b_i} \delta(\rho)^{a_1\ldots a_L b_1\ldots b_L} \prod_{k=1}^{\mathcal{L}} \Bigg[\sum_{\sigma^{(k)},\widetilde{\sigma}^{(k)}\in \mathbf{S}_{n_k}} C_N^{n_k}(\sigma^{(k)}\circ\widetilde{\sigma}^{(k)}) \times \\ \times \mathcal{T}(\sigma^{(k)})^{a_1a_2\ldots a_{n_k}}\, \mathcal{T}(\widetilde{\sigma}^{(k)})^{b_1b_2\ldots b_{n_k}} \Bigg], \tag{A.14}$$

where the index k labels the links which contribute at least one $\mathcal{U}$ in eq. (2.4), the total number of these links being $\mathcal{L}$ (note that the integral over a link carrying no $\mathcal{U}$ is 1). We call n_k the number of $\mathcal{U}$'s in (2.4) lying on the link k; as we define in section 2 the total number of $\mathcal{U}$'s to be L, it is obvious that $\sum_{k=1}^{\mathcal{L}} n_k = L$.

One can put the result in a more compact form by introducing a permutation $\vartheta \in \mathbf{S}_{2L}$ combining the set $\{\sigma^{(k)}, \widetilde{\sigma}^{(k)}; k = 1,\ldots,\mathcal{L}\}$. It is important to notice that the set $\mathcal{S}_\Gamma$ of all the possible ϑ obtained in this way is, in general, only a subgroup of $\mathbf{S}_{2L}$, determined by the structure of the joint path $\widetilde{\gamma}_1 \cup \ldots \cup \widetilde{\gamma}_n$. Notice that the condition for $C_N(\vartheta)$ to be leading in N is read from eq. (A.9) to be $\sigma^{(k)} = \widetilde{\sigma}^{(k)^{-1}}$ for any k; when written in terms of the joint permutation ϑ, this condition is equivalent to $\vartheta\tau\vartheta\tau = \mathsf{e}$, where τ is a pairing, defined in section 2, such that $\tau(a_i) = b_i, \tau(b_i) = a_i$.

Once ϑ is fixed, we can define:

$$C_N(\vartheta) \equiv 2^L \prod_{k=1}^{\mathcal{L}} C_N^{n_k}(\sigma^{(k)}\circ\widetilde{\sigma}^{(k)})\,, \tag{A.15}$$

$$\mathcal{T}(\vartheta)^{c_1\ldots c_{2L}} \equiv \prod_{k=1}^{\mathcal{L}} \mathcal{T}(\sigma^{(k)})^{a_1a_2\ldots a_{n_k}} \mathcal{T}(\widetilde{\sigma}^{(k)})^{b_1b_2\ldots b_{n_k}}\,, \tag{A.16}$$

after renaming the colour indices as in section 2. It is these definitions, together with eq. (2.6), that give the final form eq. (2.7).

B. Invariant tensor contractions at large N

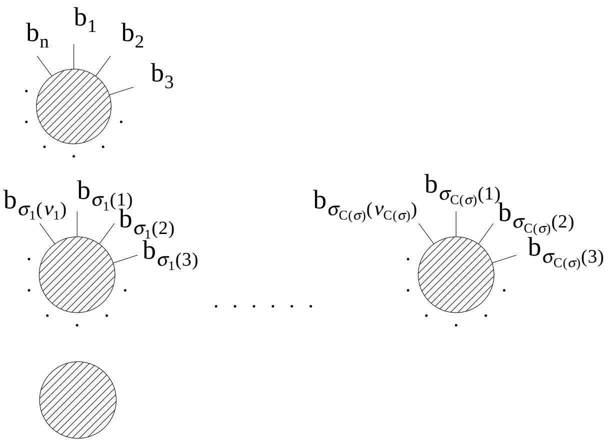
In this appendix we will study the large N behaviour of scalars (invariants) formed by contraction of invariant tensors. This type of objects appears in the definition of the quantities D_N entering in eq. (2.7). We will use the symbolic representation for tensors as given in (A.11). For notational details we refer the reader to appendix A. Computational methods for calculating these quantities have been devised [14] due to their interest in different contexts. Our iterative technique is related to the method of Cvitanović [15].

First of all we will set up a Feynman diagram-like representation for the invariant tensors $\mathcal{T}(\sigma)^{a_1\ldots a_m}$. For that, one uses the decomposition of the permutation σ into cycles (see eq. (A.13)). The tensor corresponding to an m-cycle $\mathcal{T}(\mathsf{L}_+^{(m)})^{b_1\ldots b_m}$ is the trace of the product of m group generators, as shown in eq. (A.12). Now, to each $\mathsf{L}_+^{(m)}$ tensor we associate an m-legged vertex, where the external legs are attached to indices, oriented clockwise according to the orientation given by the cycle. In this way, a tensor associated to the permutation σ is associated to $\mathcal{C}(\sigma)$ vertices. Furthermore it will prove useful to consider vertices with no legs, associated to the trace of the identity. This representation is illustrated by the following:

$$\mathcal{T}(\mathsf{L}_+^{(n)})^{b_1\ldots b_n} \longrightarrow$$

$$\mathcal{T}(\sigma)^{b_1\ldots b_n} \longrightarrow$$

$$\mathrm{Tr}(\mathbb{I}) = N \longrightarrow$$

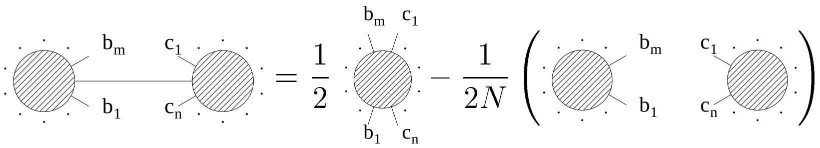


The contraction of two indices of a tensor is represented in the diagrams by joining the corresponding legs. Given a product of invariant traces with contracted indices, these prescriptions allow to represent it by a diagram consisting of a certain number of vertices, joined by internal lines, and possibly with some external lines representing unsaturated indices of the tensor. For the scalar quantities we are interested in, the corresponding diagram is a vacuum diagram with no external legs. Notice, however, that the ordering of the legs in a vertex is relevant, up to cyclic permutations. As explained in section 2 and used later, the actual pattern of contraction of indices in a tensor can be labelled by a *pairing* ρ of the indices, that is a permutation made up of only 2-cycles.

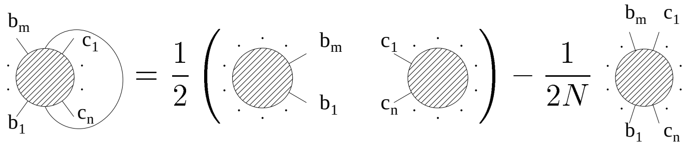
Our technique is based upon equations which express the result of contracting two indices in an $(m+n+2)$-tensor as a linear combination of $(m+n)$-tensors. These equations can be obtained easily from the completeness relation of the group generators eq. (A.3). We will refer to the basic equations as *fusion* and *fission* rules, and they are given respectively by:

$$\begin{aligned}\sum_a \mathcal{T}(\mathsf{L}_+^{(m+1)})^{ab_1\ldots b_m}\mathcal{T}(\mathsf{L}_+^{(n+1)})^{ac_1\ldots c_n} &= \frac{1}{2}\mathcal{T}(\mathsf{L}_+^{(m+n)})^{b_1\ldots b_m c_1\ldots c_n} - \\ &\quad -\frac{1}{2N}\mathcal{T}(\mathsf{L}_+^{(m)})^{b_1\ldots b_m}\mathcal{T}(\mathsf{L}_+^{(n)})^{c_1\ldots c_n} \qquad \text{(B.1)}\\ \sum_a \mathcal{T}(\mathsf{L}_+^{(m+n+2)})^{ab_1\ldots b_m a c_1\ldots c_n} &= \frac{1}{2}\mathcal{T}(\mathsf{L}_+^{(m)})^{b_1\ldots b_m}\mathcal{T}(\mathsf{L}_+^{(n)})^{c_1\ldots c_n} - \\ &\quad -\frac{1}{2N}\mathcal{T}(\mathsf{L}_+^{(m+n)})^{b_1\ldots b_m c_1\ldots c_n}\,. \qquad \text{(B.2)}\end{aligned}$$

In our diagrammatic representation the fusion and fission rules can be depicted as



(B.3)



(B.4)

It is easy to see that the repeated application of eqs. (B.3), and (B.4) allows the evaluation of all scalars formed by contracting the indices of an arbitrary tensor. Our goal will be to extract information on the N dependence from these equations. As an example and for later use, we give in table 1 the value and associated diagram for all independent tensor contractions up to six indices. The corresponding diagrams have $K = 1$ connected components and up to $L = 3$ internal lines.

If the number of indices of a tensor is odd, making all possible contractions would leave an unsaturated index. Applying the fusion-fission rules would reduce it to a linear combination of tensors with one index. But the only one-index tensor $\mathrm{Tr}(\lambda^a)$ vanishes, and hence the contribution of a diagram with a single external leg vanishes too. This allows to prove a result which will be needed in what follows: the scalar corresponding to a one-particle reducible diagram vanishes; by applying fusion and fission rules iteratively in each of those two sides of the diagram connected by the reducible line we would reach the form $B(N)\,\mathrm{Tr}(\lambda^a)\,\mathrm{Tr}(\lambda^a)$, which is zero q.e.d.

Now we will prove a result concerning the N dependence of the scalars formed by contraction of a given tensor, expressed in terms of the topological properties of the associated diagram. The result is that the leading N behaviour of such a scalar is given by $m2^{-L}N^{2K-V+L}$, with m being either 0 or 1. It is easy to check this property directly for tensors with few indices. The tensor contractions given in table 1 and their multiplication allow to check it up to $L \leq 3$. The general proof will be done by induction, making use of the fusion and fission rules. Hence, we will start by assuming that the result has been proven for all scalars obtained after the contraction of L pairs of indices, and end up proving that the result is also true for $L + 1$ contractions.

Consider then a scalar formed by performing $L+1$ contractions of the indices of a $2L+2$ invariant tensor and its associated one-particle irreducible diagram. Let say it has K connected components, V vertices and $L + 1$ internal lines. Applying the fusion rule its contribution is given as a linear combination of the contribution of two diagrams with L internal lines, with coefficients $1/2$ and $-1/2N$. The first diagram has L internal lines, $V - 1$ vertices and K connected components; thus, its leading

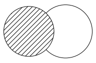
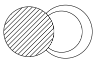
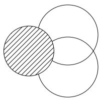
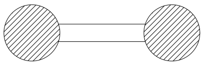
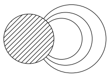
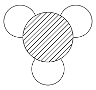
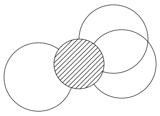
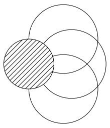
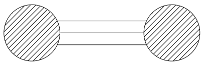
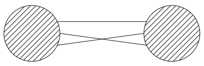
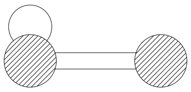
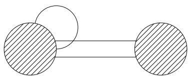
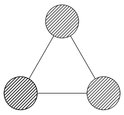
	=	$\mathrm{Tr}(\lambda^a\lambda^a)$	=	$\dfrac{N^2-1}{2}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b\lambda^b\lambda^a)$	=	$\dfrac{(N^2-1)^2}{4N}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b\lambda^a\lambda^b)$	=	$-\dfrac{N^2-1}{4N}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b)\,\mathrm{Tr}(\lambda^a\lambda^b)$	=	$\dfrac{N^2-1}{4}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b\lambda^c\lambda^c\lambda^b\lambda^a)$	=	$\dfrac{(N^2-1)^3}{8N^2}$
	=	$\mathrm{Tr}(\lambda^a\lambda^a\lambda^b\lambda^b\lambda^c\lambda^c)$	=	$\dfrac{(N^2-1)^3}{8N^2}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b\lambda^a\lambda^b\lambda^c\lambda^c)$	=	$-\dfrac{(N^2-1)^2}{8N^2}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b\lambda^c\lambda^a\lambda^b\lambda^c)$	=	$\dfrac{N^2-1}{8N^2}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)\,\mathrm{Tr}(\lambda^c\lambda^b\lambda^a)$	=	$\dfrac{(N^2-1)(N^2-2)}{8N}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)\,\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)$	=	$-\dfrac{N^2-1}{4N}$
	=	$\mathrm{Tr}(\lambda^a\lambda^a\lambda^b\lambda^c)\,\mathrm{Tr}(\lambda^b\lambda^c)$	=	$\dfrac{(N^2-1)^2}{8N}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b\lambda^a\lambda^c)\,\mathrm{Tr}(\lambda^b\lambda^c)$	=	$-\dfrac{N^2-1}{8N}$
	=	$\mathrm{Tr}(\lambda^a\lambda^b)\,\mathrm{Tr}(\lambda^b\lambda^c)\,\mathrm{Tr}(\lambda^c\lambda^a)$	=	$\dfrac{N^2-1}{8}$

Table 1: Explicit results for all topologically inequivalent, nonvanishing, completely connected diagrams corresponding to contractions of invariant traces with up to 3 internal lines are shown. Disconnected diagrams with $L \leq 3$ can be built as products of the ones shown here.

term is of the form $m_1 2^{-L} N^{2K-V+L+1}$. The second has L internal lines, V vertices and K connected components,[1] so that the leading contribution is $m_2 2^{-L} N^{2K-V+L-1}$, which is subleading with respect to the previous one. In summary, the behaviour of the initial scalar is given by $m 2^{-(L+1)} N^{2K-V+(L+1)}$, as predicted by our result, with $m = m_1$.

Now let us turn to the fission rule. As the original diagram has K connected components, V vertices and $L+1$ internal lines, then the two subdiagrams in the decomposition have $(K', V+1, L)$ and (K, V, L) topological numbers respectively, with K' being either K or $K+1$. The leading term of the original diagram is thus of the required form $m 2^{-(L+1)} N^{2K-V+(L+1)}$, with $m = 0$ if $K' = K$ and $m = m_1$ if $K' = K+1$. This completes the proof.

The previous result is used in section 2, to prove that the coefficients $D_N(\vartheta, \rho)$, which are associated to a given diagram, have the mentioned large N behaviour. This, combined with the leading behaviour of the coefficients C_N, leads to the conclusion that the large N dominant behaviour of the integrals $\mathcal{E}$ (eq. (2.3)) can only come from diagrams made out of connected components with two vertices. From our previous results we also know that the prefactor of the leading term is an integer counting the number of diagrams giving a non-vanishing maximal leading behaviour. For that purpose we need to complement our previous result by characterising those diagrams which have a maximal large N behaviour ($m = 1$). Fortunately, we only need to do this for diagrams with two vertices $V = 2$ (and no self-contractions), making up one of the connected components of a possible maximal diagram.

In the remaining of this appendix, we will prove the statement made in section 2, namely that the two-vertex connected (and with no self-contractions) diagrams having maximal leading behaviour (N^L) are the ones satisfying the identity $\vartheta\rho\vartheta\rho = \mathsf{e}$, where ϑ is the permutation giving the vertex structure and ρ the permutation specifying the contractions. By the form of the diagram we know that ϑ is made of 2 cycles of length L, and ρ of L cycles of length 2. Let us see what this means in diagrammatic terms. Take the legs of one vertex and label one as a_1, then label the subsequent legs $a_{i+1} \equiv \vartheta(a_i)$. This means that the legs are ordered in a clockwise fashion. Now call $b_i \equiv \rho(a_i)$ the indices of the other vertex (remember there are no self-contractions). The condition $\vartheta\rho\vartheta\rho = \mathsf{e}$ is equivalent to having $\vartheta(b_i) = b_{i-1}$. This means that in the other vertex the ordering of the indices is counter-clockwise. The corresponding diagram can then be drawn by joining the legs with lines which do not cross: the diagram is planar.

Using this graphical characterisation, it is easy to show that a diagram coming from $\vartheta\rho\vartheta\rho \neq \mathsf{e}$ can never have a leading term, of the form studied above, with $m = 1$. On the other hand, an exact formula can be derived for planar diagrams by iterating

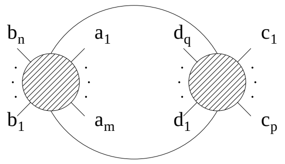
[1] It cannot have $K+1$ since this would contradict the hypothesis that the original diagram is one-particle irreducible.

the fusion rule; the result with $L \geq 2$ lines is:

$$\left(\frac{N^2-1}{2N}\right)^L \left(1 - \frac{1}{(1-N^2)^{L-1}}\right), \tag{B.5}$$

and obviously gives leading behaviour with $m = 1$.

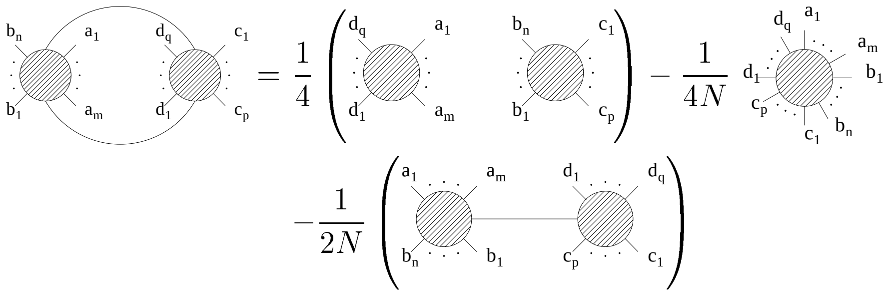
To prove that any nonplanar diagram is necessarily not leading, let us depict a generic nonplanar diagram with $L = s + 2$ lines in the following form (which is always attainable),



(B.6)

where the indexed legs are completely contracted between themselves (remember that no self-contractions are allowed), and $s = m + n = p + q \geq 1$. The nonplanarity condition enters as a constraint: both $a - c$ and $a - d$ contractions exist.

If a fusion is applied, and then a fission to the resulting first term in the r.h.s., one gets:



(B.7)

Now, the nonplanarity constraint prevents the first term in the rhs to have more than one connected component. It is then immediate, after applying the rule for the leading term to each diagram in the rhs, that the leading behaviour for the nonplanar diagram goes at most like N^{L-2}.

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